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An Example of Embedded Singular Continuous Spectrum for One-Dimensional Schrödinger Operators

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Abstract. We present a new example of a potential such that the corresponding Schrödinger operator in the halfaxis has singular continuous spectrum embedded in the absolutely continuous spectrum. The singular part is supported in an essentialy dense set. This generalizes a result of C. Remling [3].

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1. Introduction

In this Letter we will consider one-dimensional Schrödinger equation on the half-line

$$-y''(x) + V(x)y(x) = Ey(x), \quad x \in [0, \infty) \tag{1}$$

and the associated Schrödinger operators on $L_2[0, \infty)$ given by

$$(H_\alpha u)(x) = -u''(x) + V(x)u(x) \tag{2}$$

$$u(0)\cos\alpha + u'(0)\sin\alpha = 0, \quad \alpha \in [0, \pi) \tag{3}$$

There is a decomposition of the spectrum of these operators corresponding to the decomposition of a measure on $\mathbb{R}$ into pure point, absolutely continuous (respect to Lebesgue measure) and singular continuous parts. In Quantum Mechanics, in most cases, the pure point part of the spectrum is related to the bound states of the system and the absolutely continuous part to the scattering states (i.e. states which in the limit are asymptotically far from the scattering center). The remaining singular continuous part was usually supposed to admit no physical interpretation. Therefore, the study of spectrum was concentrated in giving criteria which guaranteed the absence of this kind of spectrum.

New interest in the singular continuous spectrum was generated due to the celebrated work of Pearson [2] in which, for the first time, an example of a potential with pure singular continuous spectrum and its physical interpretation were given. In this work, Pearson introduced the so-called bump potentials

$$V(x) = \sum_{n=1}^{\infty} g_n V_n(x - a_n) \tag{4}$$

where $g_n > 0$, $V_n(x) \in L_1([-B_n, B_n])$ and the intervals $[a_n - B_n, a_n + B_n]$ are disjoint. This kind of potentials lead to non-trivial asymptotics of the solutions of (1) and have been used in a number of works.

In [1] a potential with singular continuous spectrum supported on an interval I and absolutely continuous spectrum supported outside of I is given. The idea is to take a function $W(x)$, that will generate the potential, and whose Fourier transform $\widehat{W}(x)$ vanishes in a certain set S, then we have a.c. spectrum there. If $W(x)$ has compact support $\widehat{W}(x)$ cannot vanish, therefore we need a bump function of increasing support $B_n \to \infty$ such that

$$V_n(x) = \chi_{[-B_n, B_n]} W(x) \tag{5}$$

converges to $W(x)$ (see [1]).

Following this idea, Remling [3] went further giving the first example of embedded singular continuous spectrum. Using a Cantor type set, Remling constructed a bump potential so that

$$\sigma_{ac}(H_\alpha) = \sigma_{ess}(H_\alpha) = [0, \infty)$$
$$\sigma_p(H_\alpha) \cap (0, \infty) = \emptyset$$

and, for a set of boundary conditions α of positive measure, one has

$$\sigma_{sc}(H_\alpha) \cap (0, \infty) \neq \emptyset.$$

Actually, using the result of [4], it is possible to describe more precisely the location of this set of boundary conditions.

In Remling's example, the singular continuous spectrum is supported in a Cantor type set which is nowhere dense. In the present note we improve the above mentioned result giving an example of an embedded singular continuous spectrum supported in a dense set. The basic idea of the construction is the expectation that one should have sc spectrum where the Fourier transforms of the bumps are not square summable and ac spectrum elsewhere.

In order to get it, in Section 2 we construct a set $F = \bigcup_{n=0}^{\infty} F_n$ so that F and its complement F^C are essentially dense in an interval. Our potential will be the bump potential (4) and (5) with

$$W(x) = \int_{\mathbb{R}} \sum_{n=0}^{\infty} c_n \chi_{F_n}(t) \cos 2xt \, \mathrm{d}t \tag{6}$$

and $c_n > 0$, such that

$$\sum_{n=N}^{\infty} c_n \leqslant C 2^{(1-\gamma)N} \tag{7}$$

where C is a constant, $\gamma > 2$ (for example $c_n = 2^{(1-\gamma)n}$). In Section 3, we present the main ideas of the proof. These are closely related to the ones presented in [3]. A new Lemma 2 and some modifications of the proofs of Lemma 1 and Theorem 1 are needed. We also show the construction of an embedded singular continuous spectrum with dense support in the full half-line.

2. Construction of the set F

In this section, we will construct a set $F = \bigcup_{n=0}^{\infty} F_n$ so that F and its complement F^C are essentially dense in an interval $[a, b]$. Here F_n are disjoint Cantor type sets such that each F_n is constructed in the 'holes' of some previous set. So construct F_n as follows:

(1) To construct F_0, let $\delta_n > 0$ be sufficiently small prescribed numbers and $n \in \mathbb{N}$. Fix $J_0 = [a, b] \subset (0, \infty)$ and let $J_1 = J_0 \setminus (c_1^{(0)} - \delta_0, c_1^{(0)} + \delta_0)$ where $c_1^{(0)}$ is the center of J_0. In general, if J_n is a disjoint union of 2^n closed intervals with centers $c_m^{(n)}$ $(m = 1, \ldots, 2^n)$, set $J_{n+1} = J_n \setminus \bigcup_{m=1}^{2^n} (c_m^{(n)} - \delta_n, c_m^{(n)} + \delta_n)$. The set $F_0 = \bigcap_{n=1}^{\infty} J_n$ is a Cantor type set and has Lebesgue measure $|F_0| = b - a - \sum_{n=0}^{\infty} 2^{n+1}\delta_n$. We assume that δ_n are so small that $\sum_{n=0}^{\infty} 2^{n+1}\delta_n < \infty$ and $|F_0| > 0$. Denote by J_{1k} the intervals removed in the construction of F_0, so $J_{1k} = (c_m^{(n)} - \delta_n, c_m^{(n)} + \delta_n)$ for some n, m depending on k.

(2) In each interval J_{1k} construct as above a Cantor type set of positive measure F_{1k}. We assume that the corresponding δ_n^{1k} satisfy $\delta_n^{1k} \leqslant |J_{1k}|/4$ and $\delta_n^{1k} \leqslant \delta_n$. Let J_{2n} denote each one of the removed intervals.

(3) In general, in the ith step we have a succession of intervals J_{ik}. In each one of them, construct a Cantor type set of positive measure F_{ik}, taking δ_n^{ik} such that $0 < \delta_n^{ik} \leqslant \delta_n$ and $\delta_n^{ik} \leqslant |J_{ik}|/4$. The last condition guarantees that the length of the gaps in a descending sequence tends to zero. Denote by $J_{(i+1)n}$ each one of the removed intervals. In the end take the sets $F_{(2i+1)k}$ constructed in the odd steps and number them with a single index $n \in \mathbb{N}$: $F_n = F_{(2i+1)k}$ for some i, k depending on n and let $F = \bigcup_{n=0}^{\infty} F_n$. Set $G_n = F_{(2i)k}$ and $G = \bigcup_{n=0}^{\infty} G_n$.

The sets F and G are essentially dense in the interval $[a, b]$, that is for any subinterval $J \subset [a, b]$, $|F \cap J| > 0$ and the same for G. To see this let $J \subset [a, b]$ be an interval, there exist J_{ij} (i odd corresponds to F, i even to G) so that $J_{ij} \subset J$ and $|F| \geqslant |F \cap J| \geqslant |F \cap J_{ij}| \geqslant |F \cap F_{ij}| = |F_{ij}| > 0$ and the same for G.

3. Mixed Spectrum

In the sequel C will denote a constant whose actual value may change from one formula to the next.

LEMMA 1. *Let* $F, W(x)$ *be as above. Suppose* $\sup \delta_n 2^{\gamma n} < \infty$ *for some* $\gamma > 2$. *Then* $W(x) = O\left((1+|x|)^{-1+\frac{1}{\gamma}}\right)$.

Proof. *Let* $W_i(x) = \int_{\mathbb{R}} \chi_{F_i}(t) \cos 2tx \mathrm{d}t$. *It can be shown as in* [3, *Lemma* 1] *that*

$$|W_i(x)| \leqslant C\left((1+|x|)^{-1+\frac{1}{\gamma}}\right).$$

Here C *does not depend on* i, *and using this estimation,*

$$|W(x)| \leqslant \sum_{i=0}^{\infty} c_i C(1+|x|)^{-1+1/\gamma} = O((1+|x|)^{-1+1/\gamma}). \qquad \square$$

LEMMA 2. *Suppose that* $\sup \delta_n 2^{\gamma n} < \infty$ *for some* $\gamma > 2$ *and set* $f(k) = \sum_{n=0}^{\infty} c_n \chi_{G_n}(k)$. *Then there exist functions* $f_N(k) \in C_0^{\infty}(\mathbb{R})$, $f_N(k) \geqslant 0$ *such that*

$$\int_{\mathbb{R}} |f(k) - f_N(k)| \,\mathrm{d}k \leqslant C 2^{(1-\gamma)N}$$

and

$$\int_{\mathbb{R}} |f_N'(k)| \,\mathrm{d}k \leqslant C 2^{N+1}.$$

Proof. By (7) we have

$$\left| f(k) - \sum_{n=0}^{N} c_n \chi_{G_n}(k) \right| = \left| \sum_{n=N+1}^{\infty} c_n \chi_{G_n}(k) \right| \leqslant C 2^{(1-\gamma)N}.$$

For each Cantor type set G_n, $n = 1, \ldots, N$ take the set G_n^N which was obtained in the step N of its construction. G_n^N is a disjoint union of 2^N closed intervals:

$$G_n^N = \bigcup_{k=1}^{2^N} [a_k, b_k].$$

Then

$$|G_n^N \setminus G_n| \leqslant \sum_{n=N}^{\infty} 2^{n+1} \delta_n \leqslant C 2^{N(1-\gamma)}.$$

This implies that

$$\int \left| \sum_{n=0}^{N} c_n \chi_{G_n}(k) - \sum_{n=0}^{N} c_n \chi_{G_n^N}(k) \right| \mathrm{d}k \leqslant \sum_{n=0}^{N} c_n |G_n^N \setminus G_n| \leqslant C 2^{N(1-\gamma)}.$$

One can approximate each one of the characteristic functions $\chi_{G_n^N}(k)$, $n = 1, \dots, N$, which are discontinous in a_k, b_k, $k = 1, \dots, 2^N$ with functions $f_n^N(k) \in C_0^\infty(\mathbb{R})$, $0 \leqslant f_N(k) \leqslant 1$

$$\int_{\mathbb{R}} |\chi_{G_n^N}(k) - f_n^N(k)| \, \mathrm{d}k \leqslant 2^{(1-\gamma)N},$$

and such that for the derivative we have:

$$\int \left| \left(f_n^N(k) \right)' \right| \mathrm{d}k = \sum_{k=1}^{2^N} \int_{a_k-\Delta}^{a_k+\Delta} \left(f_n^N(k) \right)' \mathrm{d}k - \int_{b_k-\Delta}^{b_k+\Delta} \left(f_n^N(k) \right)' \mathrm{d}k$$

where $\Delta > 0$ is a sufficiently small number. Then

$$\begin{aligned} \int \left| \left(f_n^N(k) \right)' \right| \mathrm{d}k = \sum_{k=1}^{2^N} \Big\{ & f_n^N(a_k + \Delta) - f_n^N(a_k - \Delta) - \\ & - f_n^N(b_k + \Delta) + f_n^N(b_k - \Delta) \Big\} = 2^{N+1}. \end{aligned}$$

This implies that

$$\int_{\mathbb{R}} \left| \sum_{n=0}^{N} c_n \chi_{G_n^N}(k) - \sum_{n=0}^{N} c_n f_n^N(k) \right| \mathrm{d}k \leqslant \sum_{n=1}^{N} c_n 2^{(1-\gamma)N},$$

and

$$\int_{\mathbb{R}} \left| \left(\sum_{n=0}^{N} c_n f_n^N(k) \right)' \right| \mathrm{d}k \leqslant \sum_{n=0}^{N} c_n \int \left| \left(f_n^N(k) \right)' \right| \mathrm{d}k \leqslant C 2^{N+1}.$$

Set $f_N(k) = \sum_{n=0}^{N} c_n f_n^N(k)$. □

Let $L_n = (a_n - B_n - a_{n-1} - B_{n-1})$ (where $a_0 = B_0 = 0$) be the distance between the bumps and F the set constructed in the Section 2.

THEOREM 1. *Suppose that the set F satisfies the assumptions of Lemma* 1 *with $\gamma > 6$. Let $g_n = n^{-\frac{1}{2}}$, $B_n = n^\beta$ with $(1 - 2/\gamma)^{-1} < \beta < \gamma/8$ and assume that $n^{\beta/2\gamma} L_{n-1}/L_n \to 0$. Then the half-line Schrödinger operators H_α with potential $V(x)$ given by* (4), (5) *and* (6) *satisfy $\sigma_{ac}(H_\alpha) = \sigma_{ess}(H_\alpha) = [0, \infty)$, $\sigma_p(H_\alpha) \cap (0, \infty) = \emptyset$ and for every interval $I \subset [a, b]$ we have $\sigma_{sc}(H_\alpha) \cap I \neq \emptyset$ for a set of boundary conditions α of positive measure.*

Proof. Large parts of this proof are similar to the corresponding arguments of the proofs of [3, Theorem 3.3, 3.5]. Therefore, these parts of the proof will only be sketched.

It follows from $V(x) \to 0$, that $\sigma_{ess}(H_\alpha) = [0, \infty)$ and since the separations of barriers L_n grow fast, then (1) has no $L_2(0, \infty)$ solutions for $E > 0$.

We now show that for almost every $k \in F^C$ (we set $F^C = G \cup (\mathbb{R} \setminus [a, b])$) all solutions of (1) are bounded. Since F^C is dense everywhere in $[0, \infty)$, we conclude $\sigma_{ac}(H_\alpha) = [0, \infty)$ (by [5]).

Let $[c, d]$ be an interval in $[0, \infty)$ and let $f(k)$ be defined by

$$f(k) = \sum_{n=0}^{\infty} c_n \chi_{G_n}(k) + \chi_{[c,d]}(k) - \chi_{[a,b]}(k)$$

($[a, b]$ is the interval where the set F is constructed). Then we have $f(k) = 0$ if $k \in F$ and $f(k) > 0$ if $k \in G$ or $k \in [c, d] \setminus [a, b]$. Consider the measure $\mathrm{d}P(k) = cf(k)\mathrm{d}k$ where c is a constant such that $\int \mathrm{d}P(k) = 1$. Using Lemma 2 the proof is now completed as in [3, Theorem 3.5].

We now proceed as in the second part of the proof of [3, Theorem 3.3] in order to show that $\rho_\alpha^{ac}\left(F^2\right) = 0$. This also holds for the point part of ρ_α. So for every interval $I \subset [a, b]$ the spectral averaging formula (see e.g. [6]) becomes

$$0 < |F^2 \cap I| = \int_0^\pi \rho_\alpha(F^2 \cap I)\mathrm{d}\alpha = \int_0^\pi \rho_\alpha^{sc}(F^2 \cap I)\mathrm{d}\alpha$$

This implies $\rho_\alpha^{sc}\left(F^2 \cap I\right) > 0$ for a set of α positive measure. □

Remark 1. Using [4] we get the following condition to have singular continuous spectrum for a set of boundary conditions of positive measure in (α, β)

$$\beta - \alpha > \pi |\Lambda_0 \cap I| = |(F^2)^c \cap I|$$

where $I = \left(\frac{1}{4}, \frac{9}{4}\right)$. Observe that we can control the measure of F^2. For instance, we can make it as small (positive) as wish, so we get embedded singular continuous spectrum for many boundary conditions.

Remark 2. We note that the proof of Theorem 1 can be extended to yield more general results: set $F = \bigcup_{n=0}^{\infty} F_n$ where each F_n is a set constructed as described in Section 2 in the interval $[n, n+1], n = 0, 1, 2, \ldots$. Then F and its complement are essentially dense in $[0, \infty)$. Suppose that every F_n satisfies the assumptions of Lemma 1 with $\gamma > 6$. Let $W(x)$ be as in (6) and g_n, B_n, L_n as in Theorem 1. Then the half-line Schrödinger operators H_α with potential $V(x)$ given by equations (4), (5) and (6) satisfy $\sigma_{ac}(H_\alpha) = \sigma_{ess}(H_\alpha) = [0, \infty), \sigma_p(H_\alpha) \cap (0, \infty) = \emptyset$ and for every interval $I \subset [0, \infty)$ we have $\sigma_{sc}(H_\alpha) \cap I \neq \emptyset$ for a set of boundary condition α of positive measure.

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References

1. Kiselev, A., Last, Y. and Simon, B.: Modified Prfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators. *Comm. Math. Phys.* **194** (1998), 1–45.
2. Pearson, D.: Singular continuos measure in scattering theory. *Comm. Math. Phys.* **60** (1978), 13–36.
3. Remling, C.: Embedded singular continuous spectrum for one-dimensional Schrödinger operators. *Trans. Amer. Math. Soc.* **351** (1999), 2479–2497.
4. del Rio, R. and Tchebotareva, O.: Boundary conditions of Sturm-Liouville operators with mixed spectra. *J. Math. Anal. Appl.* **288** (2003), 518–529.
5. Simon, B.: Bounded eigenfunctions and absolutely continuous spectra for one-dimensional Schrödinger operators. *Proc. Amer. Math. Soc.* **124** (1996), 3361–3369.
6. Simon, B.: Spectral analysis of rank one perturbations and applications. *CMR Proceedings and Lecture Notes*, **8** (1995), 109–149.